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Radiation Spectrum and Correction Entropy of $(n+4)$-dimensional Kerr-(A)dS Black Hole

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Abstract We extend the classical Damour-Ruffini method and discuss Hawking radiation of an $(n+4)$-dimensional Kerr-(A)dS black hole. We not only derive the radiation spectrum that satisfies the unitary principle but also obtain the correction term of Bekenstein-Hawking entropy.

Keywords $(n+4)$-dimensional Kerr-(A)dS black hole · Energy conservation · Bekenstein-Hawking entropy correction

1 Introduction

In 1974, Hawking discovered thermal radiation of black holes [1]. This discovery not only solves the problem in black hole thermodynamics but also announces the relation among quantum mechanics, thermodynamics and gravitation, which sets a milestone in black hole physics. Investigating the physical mechanism of black hole Hawking radiation is an important subject in theoretical physics. So far there are many methods to calculate Hawking radiation, such as Hawking method [1], Damour-Ruffini method [2, 3], tunneling method proposed by Parikh and Wilczek [4], covariant abnormal method developed by Robinson and Wilczek [5]. In recent years, various methods are used to investigate thermal radiation of black holes [6–30]. They all derive the outgoing rate of black hole radiation particle is

$$\Gamma = \exp[\Delta S], \tag{1}$$

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where ΔS is Bekenstein-Hawking (BH) entropy difference before and after the black hole radiation. It satisfies the unitary principle and supports the principle of information conservation. When the black hole radiation spectra are calculated by tunneling method, the Wentzel-Kramers-Brillouin (WKB) approximate method is often adopted. However, because the wave equation is not changed to a differential equations for WKB solving, the BH entropy correction term in the calculation result is lost.

The research on the correction value of BH entropy of black holes is one of the hot researches. There are many methods to discuss the correction value of BH entropy [12, 13, 16, 19, 21, 28, 31–39]. Most people believe that the correction expression of BH entropy of Schwarzschild black hole is

$$S = \frac{A}{4G} + \chi \ln \frac{A}{4G}, \tag{2}$$

where A is the area of the black hole horizon, χ is a dimensionless constant. At present, the exact value of logarithmic term coefficient in the correction to black hole BH entropy is not clear. Correction to BH entropy of black holes in complex spacetime has not yet been reported.

In this letter, we extend the Damour-Ruffini method [2] to discuss radiation spectrum and entropy correction in Kerr-(A)dS black hole. Our analysis starts at Sect. 2 where we present the thermodynamics quantities of the $(n+4)$-dimensional Kerr-(A)dS black hole. In Sect. 3, we solve the Klein-Gorden equation with taking the reaction of the radiation of particles into consideration. The radiation spectrum and the BH entropy correction of the $(n+4)$-dimensional Kerr-(A)dS black hole were presented in Sect. 4. We finish with the presentation of our conclusions in Sect. 5.

2 Kerr-(A)dS black hole

The metric of the $(n+4)$-dimensional Kerr-(A)dS black hole for the simply rotating case [40] when written in Boyer-Lindquist coordinates is [41]

$$\begin{aligned} ds^2 = &-\frac{\Delta_r}{\rho^2}\left(dt - \frac{a}{1+\lambda a^2}\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \\ &+ \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(adt - \frac{r^2+a^2}{1+\lambda a^2}d\phi\right)^2 + r^2\cos^2\theta d\Omega_n^2, \end{aligned} \tag{3}$$

where for positive cosmological constant Λ, we have

$$\lambda = \frac{2\Lambda}{(n+2)(n+3)}, \tag{4}$$

$$\begin{aligned} &\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta_\theta = 1 + \lambda a^2\cos^2\theta, \\ &\Delta_r = (r^2+a^2)\left(1-\lambda r^2\right) - \frac{2M}{r^{n-1}}, \end{aligned} \tag{5}$$

$d\Omega_n^2$ is the metric of the n-dimensional unit sphere, which implies the location of the horizon [42]

$$\Delta_r(r_+) = 0. \tag{6}$$

Thus, we derive the relation between the black hole horizon location r_+ and the mass

$$2M = r_+^{n-1}(r_+^2 + a^2)(1 - \lambda r_+^2). \tag{7}$$

3 Klein-Gordon Equation and Tortoise Coordinate Transformation

In curved spacetime, Klein-Gordon equation of the particle with rest mass μ_0 is

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu \Phi) - \mu_0^2\Phi = 0, \tag{8}$$

where the determinant is given by the product of the base metric and higher-dimensional spherical harmonics

$$\sqrt{-g} = \left|1 + \lambda a^2\right|^{-1} \rho^2 \sin\theta \otimes r^n \cos^n\theta \prod_{i=1}^{n-1} \sin^i \theta_i. \tag{9}$$

The separation of the wave equation is implemented by making the ansatz:

$$\Phi = e^{i\omega t - im\varphi} R(r) S_{jlm}(\theta) Y_{j,i_1,i_2,\ldots,i_{n-1}}(\theta_{n-1},\phi). \tag{10}$$

In the black hole radiation research, our interest is the radial equation. From (3), the radial equation is

$$\begin{aligned}\frac{1}{r^n}\partial_r(r^n\Delta_r\partial_r R) + \Bigg[-\Lambda_{ljm} + \frac{a^2m^2}{\Delta_r}\left((1+\lambda a^2)(1-\lambda r^2) - \frac{2\lambda M}{r^{n-1}}\right) \\ - \frac{4Mam\omega}{r^{n-1}\Delta_r} + \frac{(r^2+a^2)^2\omega^2}{\Delta_r} - \frac{j(j+n-1)a^2}{r^2} - \mu_0^2\Bigg] R = 0,\end{aligned} \tag{11}$$

where ω is the energy of the radiation particles, m is the projection of the angular momentum of the radiation particle on rotation axis, Λ_{jlm} is the separation variable constant.

In order to make the radial equation turn to the form such as second order differential equation $\frac{d^2y}{dx^2} + f(x)y = 0$, which can be solved by WKB, we let

$$R = \frac{\Phi(r)}{r^{n/2}(r^2+a^2)^{1/2}}. \tag{12}$$

After the black hole radiated particles with energy ω and angular momentum m, M in spacetime line element (3) will be replaced with $M - \omega$, J be replaced with $J - m$ and Δ_r be replaced with $\Delta_{r,\omega}$. Therefore, after considering the reaction of the radiation to spacetime, we define tortoise coordinate transformation

$$dr_* = \frac{r^2 + a_\omega^2}{\Delta_{r\omega}(r, M-\omega)} dr, \tag{13}$$

where $a_\omega = \frac{J-m}{M-\omega}$. Equation (11) is reduced to

$$\frac{d^2\Phi(r)}{dr_*^2} + Q(r)\Phi(r) = 0, \tag{14}$$

where

$$Q(r)=\left[\left(\omega-\frac{am(r_\omega^2+a_\omega^2)(1-\lambda r_\omega^2)}{(r^2+a_\omega^2)^2r^{n-1}}r_\omega^{n-1}\right)^2-\frac{\Delta_{r\omega}(r)}{(r^2+a_\omega^2)^2}U(r)\right],$$

$$\begin{aligned}U(r)=\Lambda_{ljm}&-\frac{a_\omega^2m^2}{(r^2+a_\omega^2)^2}\left((r^2+a_\omega^2)(1+\lambda a_\omega^2)+\frac{(r_\omega^2+a_\omega^2)(1-\lambda r_\omega^2)}{r^{n-1}}r_\omega^{n-1}\right)\\&+\frac{n(n+2)}{4}(1-\lambda a_\omega^2)-\frac{(n+2)(n+4)}{4}\lambda r^2-\lambda a_\omega^2 eq15\\&+\left(j+\frac{n}{2}\right)\left(j+\frac{n}{2}-1\right)\frac{a_\omega^2}{r^2}+\frac{a_\omega^2(1+\lambda a_\omega^2)}{r^2+a_\omega^2}\\&+\frac{((n+2)r^2+na_\omega^2)^2-8a_\omega^2r^2}{4(r^2+a_\omega^2)^2}\frac{(r_\omega^2+a_\omega^2)(1-\lambda r_\omega^2)}{r^{n-1}}r_\omega^{n-3}-\mu_0^2\end{aligned}\tag{15}$$

and r_ω satisfies $\Delta_{r\omega}(r_\omega)=0$.

4 Radiation Spectrum and BH Entropy Correction

Based on $\Delta_{r,\omega}(r_\omega)=0$, near $r=r_\omega$ (14) is written approximately as

$$\frac{d^2\Phi(r)}{dr_*^2}+(\omega-\omega_0)^2\Phi(r)=0,\tag{16}$$

where $\omega_0=m\Omega_\omega$, $\Omega_\omega=\frac{a_\omega(1-\lambda r_\omega^2)}{r_\omega^2+a_\omega^2}$. According to the method proposed by [21], after the black hole radiates particles with energy ω and angular momentum m, on surface $r=r_\omega$ the outgoing rate is

$$\Gamma_\omega=\left|\frac{\Phi_{out}(r>r_\omega)}{\Phi_{out}(r<r_\omega)}\right|^2=e^{-4\pi(\omega-\omega_0)/\kappa_\omega},\tag{17}$$

where $\kappa_\omega=\frac{\Delta'(r_\omega)}{r_\omega^2+a_\omega^2}$. Since the process that the black hole radiates particles with energy ω and angular momentum m is an integration process [21, 43, 44], that is $\omega=\int_0^\omega d\omega'$, $m=\int_0^m dm'$. So the outgoing rate that the black hole radiates particles with energy ω and angular momentum m is

$$\Gamma(i\to f)=\prod_i\Gamma_{\omega_i}=\exp\left[-\int_0^\omega\frac{4\pi d\omega'}{\kappa_{\omega'}}+\int_0^m\frac{4\pi\Omega_{\omega'}dm'}{\kappa_{\omega'}}\right].\tag{18}$$

Thermodynamic quantities corresponding to black holes meet the first law of thermodynamics

$$dE=TdS+\Omega dJ,\tag{19}$$

we derive

$$d(\Delta S)=-\frac{d\omega'}{T_{\omega'}}+\frac{\Omega_{\omega'}}{T_{\omega'}}dm',\tag{20}$$

where

$$T_{\omega'} = \frac{(1-\lambda r_{\omega'}^2)\Delta'(r_{\omega'})}{4\pi(r_{\omega'}^2 + a_{\omega'}^2)}, \qquad \Omega_{\omega'} = \frac{a_{\omega'}(1-\lambda r_{\omega'}^2)}{r_{\omega'}^2 + a_{\omega'}^2}, \tag{21}$$

the BH entropy difference before and after the black hole radiation is

$$\Delta S = S_{\omega'}(M-\omega', J-m') - S(M, J). \tag{22}$$

Considering the reaction of the radiation of particles to the spacetime, Hawking radiation spectrum is

$$\Gamma(i \to f) = \prod_i \Gamma_{\omega_i} = \exp \int d(\Delta S) = e^{\Delta S}. \tag{23}$$

In the radiation rate calculation (17), we do not consider the factor $1/[r^{n/2}(r^2+a^2)^{1/2}]$ in (12). After we consider this factor $1/[r^{n/2}(r^2+a^2)^{1/2}]$ and the black hole radiates particles with energy ω and angular momentum m, the outgoing rate formula (23) should be written as [21]

$$\Gamma(i \to f) = \frac{r_i^n(r_i^2 + a_i^2)}{r_f^n(r_f^2 + a_f^2)} e^{\Delta S} = \exp\left[\left(\frac{A_f}{4} - \ln\frac{A_f}{4}\right) - \left(\frac{A_i}{4} - \ln\frac{A_i}{4}\right)\right]. \tag{24}$$

Thus the first order correction to BH entropy is

$$S_B = \frac{A}{4} - \ln\frac{A}{4}, \tag{25}$$

where A is the area of the black hole horizon.

Compared (25) with (2), we derive $\chi = -1$ in (2). So we obtain the correction to BH entropy of the $(n+4)$-dimensional Kerr-(A)dS black hole.

5 Conclusion

We extend the method that Damour-Ruffini has used to discuss Hawking radiation and investigate radiation spectrum of the $(n+4)$-dimensional Kerr-(A)dS black hole under the condition that the total energy is conserved and self-gravitation exists. Taking the reaction of the radiation of particles to the spacetime into consideration, we discuss the radiation spectrum of black hole by the new tortoise coordinate transformation (13). Our result is consistent with the one of Parikh and Wilczek. The radiation spectrum satisfies the unitary principle.

In our calculation, in order to make the radial equation after separation of variables change into the second-order differential equations $\frac{d^2y}{dx^2} + f(x)y = 0$ suitable for WKB solving, we introduce coordinate transformation (13). That is we introduce factor $r^{-n/2}(r^2 + a^2)^{-1/2}$ in (10), which make us derive a black hole BH entropy correction term. Most research on the black hole radiation spectra are used to obtain solutions by WKB method. In their calculation, they all neglect the factor $r^{-n/2}(r^2 + a^2)^{-1/2}$. So we have given a more comprehensive conclusion and obtained a further understanding of the black hole thermal radiation.

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